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ON CONTROL ACCURACY UNDER CONDITIONS OF PERTURBATIONS WITH BOUNDED AMPLITUDE SPECTRUM

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V. M. KEIN and A. N. PARIKOV
(Moscow)

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The problem of determining the maximum error in a linear system under the action of perturbations with bounded amplitude spectrum is analyzed. A method is suggested for selecting from a class of admissible perturbations those that are most unfavorable in the sense of the chosen accuracy criterion. An example of the determination of the maximum lateral deviation of an aircraft with an on-board control system at the final approach stage is given.

The problem of the accumulation of perturbations in a linear dynamic system was posed in [1] and its solution obtained for perturbations bounded only in modulus. A perturbation with bang-bang characteristics proved to be extremal (most unfavorable). However, in many cases such perturbations are considerably different from those actually possible, which leads to an overestimating of the maximum error. Additional constraints of a differential and an integral nature were introduced to allow for the properties of real perturbations more accurately [2, 3]. Many peculiarities of real perturbations can be accounted for by specifying constraints in the frequency domain [4]. Constraints of this kind are used below.

1. Estimate of maximum errors. Let a closed-loop dynamic system be described by the linear vector equation

$$x^{\cdot} = A(t)x + C(t)v, \quad 0 \leqslant t \leqslant T \tag{1.1}$$

Here x is the system's n-dimensional state vector, v is the m-dimensional perturbation vector, and $A(t)$ and $C(t)$ are matrices of the variable coefficients, of appropriate dimensions. Each components $v_i(t)$ of the perturbation vector can be represented by a Fourier transform

$$v_i(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty V_i(\omega) \cos[\omega t - \varphi_i(\omega)]\, d\omega \tag{1.2}$$

where the amplitude spectra $V_i(\omega)$ satisfy the contraints

$$0 \leqslant V_i(\omega) \leqslant \nu_i(\omega), \quad \int_0^\infty \nu_i(\omega)\, d\omega < \infty, \quad i = 1, \ldots, m \tag{1.3}$$

but no constraints are imposed on the functions $\varphi_i(\omega)$. We are required to find the maximum possible deviation of the system's output coordinate x_k

$$I^* = \max_v \max_{0 \leqslant t \leqslant T} x_k(t) = \max_v x_k(t^*) \tag{1.4}$$

and to construct the extremal perturbation $v^*(t)$ causing such a deviation.

Let us consider the action on the system of an elementary harmonic component of perturbation

$$dv_i(\omega) = V_i(\omega)\cos[\omega t - \varphi_i(\omega)]\,d\omega \tag{1.5}$$

The response of the output coordinate of system (1.1) to such a component at a fixed instant of time t^* can be determined by the Cauchy formula

$$dx_{ik}(\omega, t^*) = \int_0^{t^*} x^{[k]}(t^*, \tau)\, c_{[i]}(\tau) V_i(\omega)\cos[\omega\tau - \varphi_i(\omega)]\,d\omega\, d\tau \tag{1.6}$$

where $x^{[k]}(t^*, \tau)$ is the k-th row of the fundamental matrix of solutions of the homogeneous equation corresponding to Eq. (1.1) and $c_{[i]}(\tau)$ is the i-th column of matrix C.

When a perturbation of form $v_i(t) = \cos\omega t$ ($\sin\omega t$) is fed into the system, the quantity $e_{ik}(\omega, t^*)$ ($f_{ik}(\omega, t^*)$) is obtained at output x_k at instant t^*. Then expression (1.6) can be rewritten as

$$dx_{ik}(\omega, t^*) = V_i(\omega)[\cos\varphi_i(\omega)\, e_{ik}(\omega, t^*) + \sin\varphi_i(\omega)\, f_{ik}(\omega, t^*)]\,d\omega \tag{1.7}$$

The quantities $e_{ik}(\omega, t^*)$ and $f_{ik}(\omega, t^*)$ cannot simultaneously equal zero; therefore, according to the lemma on circular vectograms [5] the maximum of the expression within brackets is reached when

$$\cos\varphi_i^*(\omega) = e_{ik}(\omega, t^*)/g_{ik}(\omega, t^*), \quad \sin\varphi_i^*(\omega) = f_{ik}(\omega, t^*)/g_{ik}(\omega, t^*) \tag{1.8}$$
$$g_{ik}(\omega, t^*) = \sqrt{e_{ik}^2(\omega, t^*) + f_{ik}^2(\omega, t^*)}$$

Thus, the maximum of (1.7) is ensured by the fulfilment of conditions (1.8) and $V_i^*(\omega) = \nu_i(\omega)$. The expression for the maximum possible deviation of output x_k under the action of the i-th component of the perturbation is

$$\max dx_{ik}(\omega, t^*) = \nu_i(\omega)\, g_{ik}(\omega, t^*)\,d\omega \tag{1.9}$$

The expression for the component $v_i^*(t)$ of the extremal perturbation can be obtained by substituting the values of $V_i^*(\omega)$ and $\varphi_i^*(\omega)$ into the original expression (1.2) for the perturbation

$$\frac{dv_i^*(t)}{d\omega} = \sqrt{\frac{2}{\pi}}\,\frac{\nu_i(\omega)}{g_{ik}(\omega, t^*)}[e_{ik}(\omega, t^*)\cos\omega t + f_{ik}(\omega, t^*)\sin\omega t] \tag{1.10}$$

System (1.1) is linear and, therefore, the estimate $\max_v x_k(t^*)$ can be found by integrating expression (1.9) with respect to frequency with a subsequent summation over all components of the perturbation

$$\max_v x_k(t^*) = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{m}\int_0^\infty \nu_i(\omega)\, g_{ik}(\omega, t^*)\,d\omega \tag{1.11}$$

To determine estimate (1.4) the maximum of expression (1.1) in time usually has to be found by using numerical search methods. But sometimes the properties of system (1.1) are such that the monotonicity of the dependence of $\max_v x_k(t)$ on time can be stated explicitly. In such case the calculations are carried out only for a finite instant.

2. **Features of the computing procedure.** To compute the quantities $e_{ik}(\omega, t^*)$ and $f_{ik}(\omega, t^*)$ for various values of frequency ω_l, $l = 1, \ldots, L$ it is necessary to integrate the system of $n + 2mL$ first-order differential equations

$$ds/d\tau = A'(\tau)s, \quad s_k(0) = 1, \quad s_j(0) = 0, \quad j \neq k \tag{2.1}$$
$$\frac{de_{ik}(\omega_l, \tau)}{d\tau} = c'_{[i]}s \cos \omega_l \tau, \quad e_{ik}(\omega_l, 0) = 0$$
$$\frac{df_{ik}(\omega_l, \tau)}{d\tau} = c'_{[i]}s \sin \omega_l \tau, \quad f_{ik}(\omega_l, 0) = 0$$
$$\tau = t^* - t, \; 0 \leqslant \tau \leqslant t^*, \; i = 1, \ldots, m; \; l = 1, \ldots, L; \; j = 1, \ldots, n$$

where $s'(\tau) = x^{[k]}(t^*, \tau)$ (the prime denotes transposition). The numerical integration step for the system of Eqs. (2.1) is selected on the basis of the properties of system (1.1) and of the computation accuracy requirements. The frequency axis segment on which the integral in (1.11) is computed is determined by the form of the amplitude spectra $\nu_i(\omega)$, while the step of numerical integration with respect to frequency is selected from the formula

$$\Delta\omega \leqslant 2\pi / T \tag{2.2}$$

where T is the duration of operation of the system. When constructing the extremal perturbation $v^*(t)$, to the operations listed we add on a multiple integration of expressions of form (1.10) with respect to frequency with a step $\Delta\omega$ for discrete values of time on the interval $[0, t^*]$.

3. **Determination of maximum error in an on-board control system.** The method presented was used to determine the maximum lateral deviation of a TU-134 aircraft with an on-board control system BSU-3P [6, 7]. The segment of the final landing approach from the point of entering the glide path down to the decision taking altitude was examined, which corresponds to 90 sec of flight time at a velocity of 75 m/s. During the landing approach the lateral force necessary for moving the aircraft toward the runway axis and for compensating for the action of the cross wind W_z is created by changing the bank. The magnitude of bank γ_s needed is determined on the on-board computer from the deviation of the aircraft center of mass from the runway axis z and its rate of change $z^\cdot$. Data on these quantities are fed in from the landing radar system; distortions in the heading line of the radio beacon can cause significant errors in the determination of the angle $\Delta\varepsilon$ between the runway axis and the direction from the beacon to the aircraft. The mismatch between the current bank γ and the specified γ_s is corrected by the autopilot. The process for stabilizing the specified bank is aperiodic and, therefore, in the investigation of path control accuracy the dynamics of aircraft motion with respect to the bank are accounted for by one first-order equation. Thus, the aircraft lateral motion is characterized by the vector $x = (zz^\cdot\psi\psi^\cdot\gamma)'$, where ψ is the yaw angle relative to the runway axis. Having added to the object's equations of motion the laws for forming and executing the prescribed bank and the rudder deflection angle δ, we obtain for the closed-loop system the equation

$$x^\cdot = A(t)x + C(t)v, \quad x(0) = 0, \quad 0 \leqslant t \leqslant T \tag{3.1}$$
$$x = (zz^\cdot\psi\psi^\cdot\gamma\delta\gamma_s x_8)'$$
$$x_8 = \gamma_s^\cdot + 4.116 i_\varepsilon \Delta\varepsilon$$

The perturbation vector $v = (\Delta\varepsilon, W_z)'$ accounts for the action of the "force" W_z and information interference $\Delta\varepsilon$. The matrices A and C have the form

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0762 & -5.34 & 0 & 9.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.0056 & -0.392 & -0.0889 & -0.0378 & -0.17 & 0.0378 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -0.0129 & -0.902 & -0.2045 & -0.0869 & -0.89 & 0.0869 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_{81}(t) & a_{82}(t) & 0 & 0.354 & 0 & 0 & -0.253 & -1.01 \end{Vmatrix}$$

$$C = \begin{Vmatrix} 0 & 0 \\ 0 & 0.0762 \\ 0 & 0 \\ 0 & 0.0056 \\ 0 & 0 \\ 0 & 0.012 \\ -4.116 & 0 \\ 3.905 & 0 \end{Vmatrix}$$

The coefficients a_{81} and a_{82} depend upon the distance between the aircraft and the localizer beacon and, for constant flight velocity, vary according to the laws

$$a_{81}(\tau) = -\frac{0.00361 i_\varepsilon}{72.14+\tau}\left(1+\frac{16.3}{72.14+\tau}\right) \tag{3.2}$$

$$a_{82}(\tau) = -\frac{0.0588 i_\varepsilon}{72.14+\tau}, \quad \tau = T - t, \quad T = 90\text{ s}$$

$$i_\varepsilon = \begin{cases} 4.68, & \tau < 40.7\text{ s} \\ 5.85, & \tau \geqslant 40.7\text{ s} \end{cases}$$

where i_ε is the computer's transfer constant. For the flight mode being examined the object's state at the final instant of time is of the utmost importance; therefore, criterion (1.4) was adopted as the terminal criterion

$$I^* = \max_v x_1(T) = \max_v z(T) \tag{3.3}$$

The amplitude spectra $\nu_i(\omega)$ of the perturbations being analyzed were specified, in accord with [7, 8], in the form (see Fig. 1): for distortions in the heading line with due regard to only the low-frequency component of $\Delta\varepsilon$

$$\nu_\varepsilon(\omega, \tau) = b_\varepsilon(\tau) f_\varepsilon(\omega) \tag{3.4}$$

$$b_\varepsilon(\tau) = b_\varepsilon^\circ + \xi_\varepsilon \tau, \quad b_\varepsilon^\circ = 0.0023 \text{ rad}$$

$$\xi_\varepsilon = 0.000125 \text{ rad/s}$$

$$f_\varepsilon(\omega) = \left(\frac{0.032+0.4\omega^2}{0.0269+4.2\omega^4}\right)^{1/2}$$

for the velocity of the cross wind W_z

$$\nu_W(\omega, \tau) = b_W(\tau) f_W(\omega) \tag{3.5}$$

$$b_W(\tau) = b_W^\circ + \xi_W \tau, \quad b_W^\circ = 30.8 \text{ m/s} \quad \xi_W = 0.314 \text{ m/s}^2$$

$$f_W(\omega) = \frac{(0.102+1.4\omega^2)^{1/2}}{0.438+2.013\omega^2}$$

With due regard to the inertia of the object being examined the frequency range to be accounted for when estimating the maximum deviation was restricted to $\omega_m = 2$ Hz. In accord with (2.2) the step of numerical integration with respect to frequency was $\Delta\omega = 0.069$ Hz. The computation of estimate (3.3) required the integration of a system of $8 + (2 \times 2 \times 29) = 124$ equations on the interval [0, 90 s] with a step of 0,1 s followed by an integration of expression (1.9) on the frequency interval [0, 2 Hz] with step $\Delta\omega$. The maximum error in system (3.1) with respect to lateral deviation at the decision altitude was 22.16 m. The values of the components of the extremal perturbation $v^*(t)$, $t \in [0, 90$ s], with a step of 1 s (Fig. 2) were obtained by intergrating $2 \times 91 = 182$ equations of form (1.10) on the interval [0, 2 Hz] with step $\Delta\omega$. The system's motion was simulated for the action of extreme distortions of the heading line (Fig. 3, curve 1), of cross wind (curve 2), and under the combined action of the extremal perturbations (curve 3). In the last case the maximum deviation at the final instant was 21.37 m. The insignificant discrepancy between the values of the maximum error (3.5%) as obtained from formula (1.11) and from the simulation is explained by the imprecise representation of perturbation $v^*(t)$ within the intervals of 1 s.

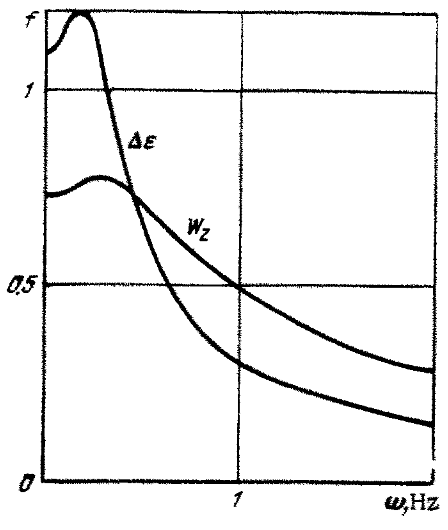

Fig. 1

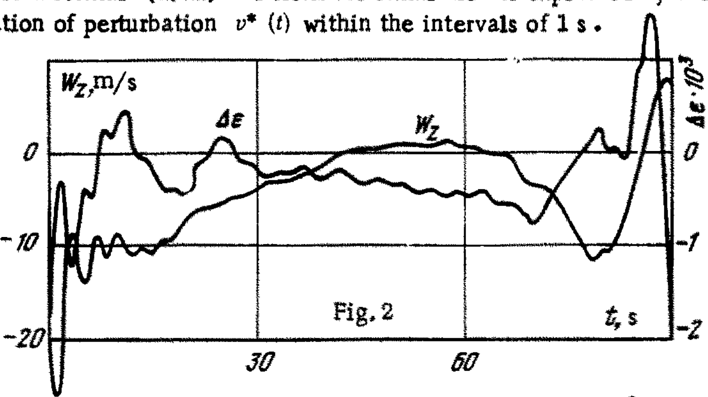

Fig. 2

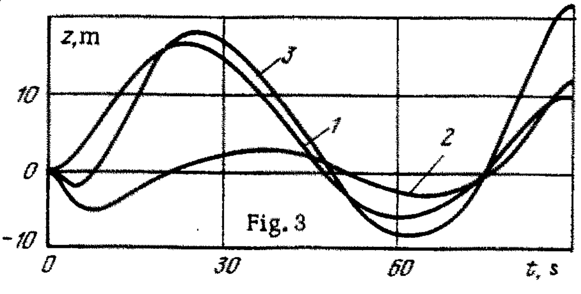

Fig. 3

The program run time for determining the maximum error (3.3), for the construction of the extremal perturbation $v^*(t)$, and for simulating the motion of system (3.1) was 3 min 30 sec on the electronic computer.

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